Linear Algebra II

03/04/2018, Tuesday, 14:00 – 17:00

Please write your name and student number on the exam ánd on the envelope. The exam contains 6 problems.

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1 $(8 + 7 = 15 \text{ pts})$ **Orthonormal basis**

Consider the vector space $\mathbb{R}^4$ with the inner product

$$\langle x, y \rangle = x^T y.$$

Let $S \subset \mathbb{R}^4$ be the subspace given by

$$S = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(a) Determine an orthonormal basis for S.

(b) Find the closest element in the subspace S to the vector

$$\begin{bmatrix} a \\ b \\ a \\ b \end{bmatrix}$$

where a and b are real numbers.

2 $(8 + 7 = 15 \text{ pts})$ **Cayley-Hamilton**

(a) For given real numbers a, b, c, d we consider the real matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determine α and β such that $A^2 + \alpha A + \beta I = 0$.

(b) For given real numbers a, b, c, d we consider the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & -a \\ 1 & 0 & 0 & -b \\ 0 & 1 & 0 & -c \\ 0 & 0 & 1 & -d \end{bmatrix}$$

Determine α, β, γ and δ such that $A^4 + \alpha A^3 + \beta A^2 + \gamma A + \delta I = 0$.

3 $(3+8+4=15$ pts) **Singular value decomposition**

Consider the matrix

$$M = \begin{bmatrix} 6 & 2 \\ -7 & 6 \end{bmatrix}.$$

(a) Show that the singular values of M are 10 and 5.

(b) Find a singular value decomposition for M.

(c) Find the best rank 1 approximation of M.

4 $(5+5+5=15$ pts) **Positive semi-definite matrices and eigenvalues**

Let A be a real symmetric $n \times n$ matrix.

(a) Prove that all eigenvalues of A are real.

(b) Prove that if A is positive semidefinite then every eigenvalue λ of A satisfies $\lambda \geqslant 0$.

(c) Prove that if every eigenvalue λ of A satisfies $\lambda \geqslant 0$ then A is positive semidefinite.

5 $(3+2+3+3+4=15$ pts) **Eigenvalues and multiplicity**

(a) Let A be a real $m \times n$ matrix and B a real $n \times m$ matrix. Let $\lambda \neq 0$. Show that λ is an eigenvalue of AB if and only if it is an eigenvalue of BA.

Next, let a and b be vectors in $\mathbb{R}^n$ such that $a^T b \neq 0$.

(b) Show that $a^T b$ is an eigenvalue of the $n \times n$ matrix $b\,a^T$.

(c) Show that $R(b\,a^T)$ is equal to $R(b)$.

(d) Determine $\text{rank}(b\,a^T)$

(e) Show that 0 is an eigenvalue of $b\,a^T$ and determine its geometric multiplicity.

6 $(5+5+5=15$ pts) **Jordan canonical form**

Consider the matrix

$$M = \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 0 & b \end{bmatrix}$$

where a and b are arbitrary real constants.

(a) Determine for all a and b the eigenvalues of M together with their algebraic multiplicities.

(b) Give necessary and sufficient conditions on a and b under which M is in Jordan canonical form

(c) Assume that $a \neq b$. Determine the Jordan canonical form of M. Explain your answer clearly!

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10 pts free